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Ideals and Filters in D-Posets

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Ideals and filters in D-posets are defined. The role of idempotent elements and atoms in the theory of ideals and filters is studied.

1. INTRODUCTION

D-posets, as they were defined by Kôpka and Chovanec (1994), generalize orthomodular posets, MV algebras, and orthoalgebras. Foulis and Bennett (1994) have shown the equivalence between D-posets and effect algebras. Ideals and filters were defined in all these structures (Chang, 1958; Foulis, *et al.*, 1992). A more general definition of an ideal is given in Chovanec and Kôpka (1996).

The aim of this paper is to study other properties of ideals and filters in D-posets. From the theory of orthomodular posets it is known that every interval $[0, a]$, $a \neq 1$, is a proper ideal. We give a necessary and sufficient condition for an interval $[0, a]$ to be a proper ideal in a D-lattice. The last theorem provides a sufficient condition for the existence of a state on a Boolean D-poset.

2. DIFFERENCE POSETS

Let $(\mathcal{P}, \leq)$ be a nonempty, partially ordered set (a poset). A partial binary operation $\ominus$ is said to be *a difference* on $\mathcal{P}$, if an element $b \ominus a$ is defined in $\mathcal{P}$ if and only if $a \leq b$, and the following conditions are satisfied:

(D1) If $a \leq b$, then $b \ominus a \leq b$.
(D2) If $a \leq b$, then $b \ominus (b \ominus a) = a$.
(D3) If $a \leq b \leq c$, then $c \ominus b \leq c \ominus a$.

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(D4) If $a \le b \le c$, then $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

A structure $(\mathcal{P}, \le, \ominus)$ is called *a poset with a difference.* If $\mathcal{P}$ is a lattice, then $(\mathcal{P}, \wedge, \vee, \ominus)$ is called *a lattice with a difference.*

A bounded poset (a bounded lattice) with a difference is called *a difference poset* (*a difference lattice*) or briefly *a D-poset* (*a D-lattice*) (Kôpka and Chovanec, 1994).

The greatest element of a D-poset $\mathcal{P}$ is denoted by $1_{\mathcal{P}}$ and the least one by $0_{\mathcal{P}}$. Obviously $0_{\mathcal{P}} = a \ominus a$ for any $a \in \mathcal{P}$.

It is possible to define a unary operation $\perp$ and a partial binary operation $\oplus$ on a D-poset $\mathcal{P}$ by

$$a^{\perp} = 1_{\mathcal{P}} \ominus a$$

and

$$a \oplus b = (a^{\perp} \ominus b)^{\perp} \qquad \text{for} \quad b < a^{\perp}$$

If $\mathcal{P}$ is a D-lattice, then there exists an extension of a partial binary operation $\ominus$ on a (total) binary operation "$-$" defined as follows:

$$a - b = a \ominus (a \wedge b) \qquad \text{for every} \quad a, b \in \mathcal{P}$$

so for the dual operation "$+$" we have

$$a + b = (a^{\perp} - b)^{\perp} \qquad \text{for every} \quad a, b \in \mathcal{P} \tag{2.1}$$

The compatibility in D-posets was introduced by Kôpka (1995). Two elements $a, b \in \mathcal{P}$ are compatible in $\mathcal{P}$ (denoted by $a \leftrightarrow b$) if there exist elements $c, d \in \mathcal{P}$ such that $d \le a \le c$, $d \le b \le c$, and $c \ominus a = b \ominus d$ (equivalently $c \ominus b = a \ominus d$).

By Chovanec and Kôpka (1995), a and b are compatible in a D-lattice $\mathcal{P}$ if and only if $(a \vee b) \ominus a = b \ominus (a \wedge b)$ [equivalently $(a \vee b) \ominus b = a \ominus (a \wedge b)$].

Kôpka (1995) defined *a Boolean D-poset* as a poset $\mathcal{P}$ with the least element $0_{\mathcal{P}}$, the greatest element $1_{\mathcal{P}}$, and with a binary operation "$-$" on $\mathcal{P}$ satisfying the following conditions:

(BD1) $a - 0_{\mathcal{P}} = a$ for any $a \in \mathcal{P}$.
(BD2) $a - (a - b) = b - (b - a)$ for every $a, b \in \mathcal{P}$.
(BD3) $a \le b$ implies $c - b \le c - a$ for any $c \in \mathcal{P}$.
(BD4) $(a - b) - c = (a - c) - b$ for every $a, b, c \in \mathcal{P}$.

Boolean D-posets are characterized in the following theorem (Chovanec and Kôpka, 1995, 1996).

Theorem 2.1. The following assertions are equivalent:
(i) $\mathcal{P}$ is a Boolean D-poset.
(ii) $\mathcal{P}$ is a D-lattice of pairwise compatible elements.

(iii) $\mathcal{P}$ is a D-lattice with a binary difference operation "$-$" such that $(a - b) - c = (a - c) - b$.

We note that a Boolean D-poset is an MV algebra (introduced by Chang, 1958) and vice versa, an MV algebra can be organized as a Boolean D-poset (Chovanec and Kôpka, 1996).

3. Ideals and Filters in D-Posets

Definition 3.1. Let $\mathcal{P}$ be a D-poset. A nonempty subset $\mathcal{I} \subseteq \mathcal{P}$ is called *an ideal* (in $\mathcal{P}$) if:

(I1) $a \in \mathcal{I}$, $b \in \mathcal{P}$, $b \leq a$ implies $b \in \mathcal{I}$.
(I2) $a \in \mathcal{I}$, $b \in \mathcal{P}$, $a \leq b$, and $b \ominus a \in \mathcal{I}$ implies $b \in \mathcal{I}$.

A nonempty subset $\mathcal{F} \subseteq \mathcal{P}$ is called *a filter* (in $\mathcal{P}$) if:

(F1) $a \in \mathcal{F}$, $b \in \mathcal{P}$, $a \leq b$ implies $b \in \mathcal{F}$.
(F2) $a \in \mathcal{F}$, $b \in \mathcal{P}$, $b \leq a$, and $(a \ominus b)^{\perp} \in \mathcal{F}$ implies $b \in \mathcal{F}$.

An ideal $\mathcal{I}$ (a filter $\mathcal{F}$) is *proper* if $1_{\mathcal{P}} \notin \mathcal{I}$ $(0_p \notin \mathcal{F})$.
We note that the condition (I2) can be replaced by the equivalent one:

(I3) $a \in \mathcal{I}$, $b \in \mathcal{I}$, $a \leq b^{\perp}$ implies $a \oplus b \in \mathcal{I}$.

Moreover, it is evident that:

(i) $0_{\mathcal{P}} \in \mathcal{I}$ for any ideal $\mathcal{I}$ in $\mathcal{P}$.
(ii) If $\{\mathcal{I}_t$: $t \in T$, T is an index set$\}$ is a system of ideals in $\mathcal{P}$, then $\bigcap_{t\in T} \mathcal{I}_t$ is an ideal in $\mathcal{P}$, too.
(iii) $\{0_{\mathcal{P}}\}$ is the least ideal in $\mathcal{P}$.

It is easy to prove that if $\mathcal{I}$ $(\mathcal{F})$ is a proper ideal (a proper filter) and $a \in \mathcal{I}$ $(a \in \mathcal{F})$, then $a^{\perp} \notin \mathcal{I}$ $(a^{\perp} \notin \mathcal{F})$.

Proposition 3.2. Let $\mathcal{I}$ be a proper ideal and $\mathcal{F}$ be a proper filter in a D-poset $\mathcal{P}$. Then $\mathcal{I}^{\perp} = \{a^{\perp}: a \in \mathcal{I}\}$ is a proper filter and $\mathcal{F}^{\perp} = \{a^{\perp}: a \in \mathcal{F}\}$ is a proper ideal in $\mathcal{P}$.

Proof. If $a \in \mathcal{I}^{\perp}$, $b \in \mathcal{P}$, and $a \leq b$, then $a^{\perp} \in \mathcal{I}$ and $b^{\perp} \leq a^{\perp}$, which gives that $b^{\perp} \in \mathcal{I}$ and therefore, $b = (b^{\perp})^{\perp} \in \mathcal{I}^{\perp}$. Let $a \in \mathcal{I}^{\perp}$, $b \in \mathcal{P}$, $b \leq a$, and $(a \ominus b)^{\perp} \in \mathcal{I}^{\perp}$. Then $a^{\perp} \in \mathcal{I}$, $a^{\perp} \leq b^{\perp}$, and $a \ominus b = b^{\perp} \ominus a^{\perp} \in \mathcal{I}$, which implies $b^{\perp} \in \mathcal{I}$; therefore $b \in \mathcal{I}^{\perp}$. The proof of the fact that $\mathcal{F}^{\perp}$ is a proper ideal is analogous. ■

Example 3.3. Let w be a morphism of D-posets $\mathcal{P}$ and $\mathcal{T}$, i.e., $w: \mathcal{P} \to \mathcal{T}$ is a mapping such that:

(i) $w(1_{\mathcal{P}}) = 1_{\mathcal{T}}$.
(ii) $a, b \in \mathcal{P}$, $a \leq_{\mathcal{P}} b$ implies $w(a) \leq_{\mathcal{T}} w(b)$.
(iii) $a, b \in \mathcal{P}$, $a \leq_{\mathcal{P}} b$ implies $w(b, \ominus_{,\mathcal{P}} a) = w(b) \ominus_{\mathcal{T}} w(a)$.

Then the kernel $\mathcal{K}$ of a morphism w, i.e., the set $\mathcal{K} = \{a \in \mathcal{P}: w(\mathrm{a}) = 0_{\mathcal{T}}\}$, is a proper ideal in $\mathcal{P}$, and, dually, the set $\mathcal{G} = \{a \in \mathcal{P}: w(a) = 1_{\mathcal{T}}\}$, is a proper filter in $\mathcal{P}$.

Let $a \in \mathcal{P}$, $a \neq 1_{\mathcal{P}}$. Let us denote $[0_{\mathcal{P}}, a] = \{b \in \mathcal{P}: 0_{\mathcal{P}} \leq b \leq a\}$. If $\mathcal{P}$ is an orthomodular poset, then $[0_{\mathcal{P}}, a]$ is a proper ideal. This is not true, in general.

Example 3.4. Let $[0, 1]$ be an interval of real numbers with the usual difference of reals. Then $[0, 1]$ is a D-poset, even a Boolean D-poset, and $\{0\}$ is the only proper ideal in $[0, 1]$.

Now we present a necessary and sufficient condition for an interval $[0_{\mathcal{P}}, a]$, $a \neq 1_{\mathcal{P}}$, to be a proper ideal in a D-lattice $\mathcal{P}$.

First we need a notion of an idempotent element in a D-lattice.

Definition 3.5. An element a of a D-lattice $\mathcal{P}$ is said to be *an idempotent element* if

$$a + a = a$$

where "+" is the binary operation defined by (2.1).

Proposition 3.6. An element a of a D-lattice $\mathcal{P}$ is idempotent if and only if $a \wedge a^{\perp} = 0_{\mathcal{P}}$.

Proof. If $a \in \mathcal{P}$ is idempotent, then $a = a + a = (a^{\perp} - a)^{\perp} = (a^{\perp} \ominus (a^{\perp} \wedge a))^{\perp}$, which results in $a^{\perp} = a^{\perp} \ominus (a^{\perp} \ominus a)$. Hence

$$0_{\mathcal{P}} = a^{\perp} - a^{\perp} = a^{\perp} \ominus (a^{\perp} \ominus (a^{\perp} \wedge a)) = a^{\perp} \wedge a$$

Conversely, if $a \wedge a^{\perp} = 0_{\mathcal{P}}$; then

$$a + a = (a^{\perp} - a)^{\perp} = (a^{\perp} \ominus (a \wedge a^{\perp}))^{\perp} = (a^{\perp})^{\perp} = a$$

It is obvious that any element in an orthomodular lattice $\mathcal{L}$ is idempotent, because $a \wedge a^{\perp} = 0_{\mathcal{L}}$ for any $a \in \mathcal{L}$.

Theorem 3.7. Let $\mathcal{P}$ be a D-lattice, $a \in \mathcal{P}$, $a \neq 1_{\mathcal{P}}$. Then $[0_{\mathcal{P}}, a]$ is a proper ideal in $\mathcal{P}$ if and only if a is an idempotent element.

Proof. Let $[0_{\mathcal{P}}, a]$ be a proper ideal. It is evident that $a \leq a + a = (a^{\perp} \ominus (a \wedge a^{\perp}))^{\perp}$.

Then

$$(a + a) \ominus a = (a^{\perp} \ominus (a \wedge a^{\perp}))^{\perp} \ominus a$$
$$= a^{\perp} \ominus (a^{\perp} \ominus (a \wedge a^{\perp})) = a \wedge a^{\perp} \leq a$$

and by (I2), $a + a \in [0_{\mathcal{P}}, a]$, which implies $a + a \leq a$, therefore $a = a + a$.

Now, let $a = a + a$. If $b \in \mathcal{P}$, $b \leq a$, then $b \in [0_{\mathcal{P}}, a]$. If $a \leq b$ and $b \ominus a \in [0_{\mathcal{P}}, a]$, then $b \ominus a \leq 1_{\mathcal{P}} \ominus a = a^{\perp}$ and $b \ominus a \leq a$; therefore $b \ominus a \leq a \wedge a^{\perp}$. It holds that $a^{\perp} \ominus (a \wedge a^{\perp}) \leq a^{\perp} \ominus (b \ominus a)$ and $b = (1_{\mathcal{P}} \ominus b)^{\perp} = ((1_{\mathcal{P}} \ominus a) \ominus (b \ominus a))^{\perp} \leq (a^{\perp} \ominus (a \wedge a^{\perp}))^{\perp} = a + a = a$, which implies $b \in [0_{\mathcal{P}}, a]$. ■

Proposition 3.8. Let $\mathcal{P}$ be a D-lattice, $a \in \mathcal{P}$, $a \neq 1_{\mathcal{P}}$. Then $[0_{\mathcal{P}}, a]$ is a proper ideal in $\mathcal{P}$ if and only if $[0_{\mathcal{P}}, a^{\perp}]$ is a proper ideal in $\mathcal{P}$.

Proof. This proof follows immediately from the fact that $a + a = a$ if and only if $a^{\perp} + a^{\perp} = a^{\perp}$. ■

Definition 3.9. Let $\mathcal{P}$ be a D-poset. An element $a \in \mathcal{P}$ is called *an atom* (in $\mathcal{P}$) if the following implication holds:

$$\text{if } b \in \mathcal{P} \text{ and } b \leq a, \quad \text{then } b = 0_{\mathcal{P}} \text{ or } b = a$$

Theorem 3.10. Let $\mathcal{P}$ be a Boolean D-poset. If $a \in \mathcal{P}$, $a \neq 1_{\mathcal{P}}$, and a is an idempotent atom in $\mathcal{P}$, then there exists a proper ideal $\mathcal{I}$ and a proper filter $\mathcal{F}$ such that $\mathcal{I} \cap \mathcal{F} = \emptyset$ and $\mathcal{I} \cup \mathcal{F} = \mathcal{P}$.

Proof. Let us put $\mathcal{I} = [0_{\mathcal{P}}, a^{\perp}]$ and $\mathcal{F} = [a, 1_{\mathcal{P}}]$. If $x \in \mathcal{I} \cap \mathcal{F}$, then $x \leq a^{\perp}$ and $a \leq x$, which gives that $x^{\perp} \leq a^{\perp}$; therefore $x^{\perp} \in \mathcal{I}$. This contradicts the fact that $\mathcal{I}$ is a proper ideal.

Let $x \in \mathcal{P}$, $x \notin \mathcal{F} = [a, 1_{\mathcal{P}}]$. Let us calculate

$$(a^{\perp} + x) \ominus a^{\perp} = (a - x)^{\perp} \ominus a^{\perp} = (a \ominus (a \wedge x))^{\perp} \ominus a^{\perp}$$
$$= a \ominus (a \ominus (a \wedge x)) = a \wedge x$$

Since a is an atom, there are two possibilities:

$$(1)\ a \wedge x = 0_{\mathcal{P}}, \qquad (2)\ a \wedge x = a$$

If $a \wedge x = 0_{\mathcal{P}}$, then $a^{\perp} = a^{\perp} + x \geq x$, which entails $x \in [0_{\mathcal{P}}, a^{\perp}]$. If $a \wedge x = a$, then $a \leq x$, i.e., $x \in [a, 1_{\mathcal{P}}]$, which contradicts the assumption.

Similarly, if $x \in \mathcal{P}$, $x \notin \mathcal{I}$, then $x \in \mathcal{F}$. ■

A state on a D-poset was defined by Kôpka and Chovanec (1994).

Definition 3.11. A mapping $s: \mathcal{P} \to [0, 1]$ is said to be *a state on a D-poset* $\mathcal{P}$ if:

(S1) $s(1_{\mathcal{P}}) = 1$.
(S2) If $a, b \in \mathcal{P}$, $a \le b$, then $s(b \ominus a) = s(b) - s(a)$.
(S3) If $a, a_n \in \mathcal{P}$ $(n = 1, 2, \ldots)$, $a_n \le a_{n+1}$, for all $n = 1, 2, \ldots$, $a_n \nearrow a$ and $a = \vee_{n=1}^{\infty} a_n$, then $s(a_n) \nearrow s(a)$

Theorem 3.12. If $\mathcal{P}$ is a Boolean D-poset satisfying assumptions of Theorem 3.10, then there exists a state on $\mathcal{P}$.

Proof. Let $\mathcal{I} = [0_{\mathcal{P}}, a^{\perp}]$ be an ideal and $\mathcal{F} = [a, 1_{\mathcal{P}}]$ be a filter in $\mathcal{P}$. It suffices to put

$$s(x) = \begin{cases} 0 & \text{if } x \in \mathcal{I} \\ 1 & \text{if } x \in \mathcal{F} \end{cases}$$

for any $x \in \mathcal{P}$. Then s is a two-valued state on $\mathcal{P}$. ■

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